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Kac-Moody extensions of 3-algebras and $M2$-branes

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Abstract: We study the 3-algebraic structure involved in the recently shown M2-branes worldvolume gauge theories. We first extend an arbitrary finite dimensional 3-algebra into an infinite dimensional 3-algebra by adding a mode number to each generator. A unique central charge in the algebra of gauge transformations appears naturally in this extension. We present an infinite dimensional extended 3-algebra with a general metric and also a different extension with a Lorentzian metric. We then study ordinary finite dimensional 3-algebras with different signatures of the metric, focusing on the cases with a negative eigenvalue and the cases with a zero eigenvalue. In the latter cases we present a new algebra, whose corresponding theory is a decoupled abelian gauge theory together with a free theory with global gauge symmetry, and there is no negative kinetic term from this algebra.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence, Gauge Symmetry, M-Theory.

Contents

1. Introduction

Recently it has been shown that a 3-algebraic structure is relevant for the supersymmetry and gauge symmetry transformations [1 − 3] of the worldvolume theory of multiple coincident M2-branes. A candidate Lagrangian description of this theory has been found after obtaining on-shell equations of motion, arising from demanding the on-shell closure of the supersymmetry algebra. Earlier work on the non-propagating nature of the gauge fields in this theory includes the conjecture in [4] in which a Chern-Simons type self-coupling of the gauge fields was proposed to be part of the dynamics in the multiple M2-brane theory. This type of couplings not only does not introduce new independent degrees of freedom, but also has the right conformal dimension in three dimensions. The 3-algebraic structure of this theory has also been hinted at by the early study of a system of M2-branes ending on M5 [5], in which a Nambu-Poisson type 3-bracket [6] played an important role and had the ingredient for making all the transverse scalars on equal footing. The complete Lagrangian with all the requisite symmetries and a 3-algebraic gauge symmetry including a particular $so(4)$ example were found in the illuminating work of [1 − 3], and the Lagrangian theory has recently been studied from various perspectives [7]–[42].

In order to better understand the nature of the algebraic structure of the fields on the worldvolume of multiple M2-branes, in this paper, we study the 3-algebraic structure itself. We extend the 3-algebraic structure considered in [1 − 3, 7] in two directions. In the first direction, we make extensions of the finite dimensional 3-algebras into infinite dimensional 3-algebras by adding a mode number to each generator. This extension is similar in spirit to the Kac-Moody extension of Lie algebras. A central charge naturally appears on the right hand side of the algebraic relations. We also present a different extension for the Lorentzian 3-algebras [23 – 25]. This extension may be relevant to the fields on M2-branes if they are also valued in an internal circle, which may be viewed as the boundary circle of open membranes stretching between M2-branes. In the second direction, we explore the finite dimensional 3-algebra, but with different signatures of the metric for the generators, with the motivation of embedding general Lie algebras, and present a very simple 3-algebra with a zero eigenvalue in the metric. However, the gauge theory for this simple algebra is not very appealing and is a decoupled abelian gauge theory together with a theory with global symmetries of an arbitrary Lie algebra, from the point of view of the Lagrangian.

The organization of this paper is as follows. In section 2, we focus on the derivation of the extensions of the 3-algebras into infinite dimensional ones with mode numbers. In section 3, we explore 3-algebras with different metric signatures, and related Lagrangian theories. In subsection 3.1, after revisiting the derivation of the algebras with a negative eigenvalue in the metric, independently obtained by [23 – 26], we discuss 3-algebras with a zero eigenvalue in the metric. In subsection 3.2, we study Lagrangians before contracting with the metric and study the theory corresponding to the algebra with a zero eigenvalue in the metric. We also emphasize the study of a Lagrangian 2-tensor which falls in the algebra of gauge transformations. In section 4, we make brief conclusions and discuss mass deformed theories and related work.

2. Extensions and infinite dimensional algebras

2.1 Infinite dimensional extensions with general metrics

The algebras in [1, 2] and [3] are intimately connected with each other. As was speculated in [3], the scalars and spinors on the M2-branes may live in an algebra $\mathcal{A}$, and the gauge fields may live in a possibly different algebra $\mathcal{B}$. In the case of a conventional Yang-Mills theory, these two algebras are the same. A natural generalized possibility for M2-branes is that these two may not be the same. Thereby there are 3 types of 2-brackets, as formulated in [3]:

$$\langle \cdot\,,\ \cdot \rangle :\ \mathcal{A} \otimes \mathcal{A} \to \mathcal{B} \tag{2.1}$$
$$(\cdot\,,\ \cdot) :\ \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \tag{2.2}$$
$$[\cdot\,,\ \cdot] :\ \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \tag{2.3}$$

The first bracket (2.1) means that we can form an antisymmetric product of two elements in $\mathcal{A}$ to obtain a gauge transformation. The second bracket (2.2) is a gauge transformation of the element in $\mathcal{A}$ by the action of an element in $\mathcal{B}$. The third bracket (2.3) means that applying two gauge transformations is again a gauge transformation. It may be worth mentioning that if the first product (2.1) is a symmetric product instead of an antisymmetric product, then the algebra may be viewed as a super Lie algebra. However it is not, since the first bracket is an antisymmetric product, and is not a symmetric product.

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If we make the combined operation of the first bracket (2.1) and the second bracket (2.2), we obtain the 3-bracket formulated in [1, 2] as well as [3]:

$$[\cdot\,,\ \cdot\,,\ \cdot] = (\langle\cdot\,,\ \cdot\rangle,\ \cdot) :\ \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\to\mathcal{A} \tag{2.4}$$

In principle, we may also consider the case that some elements in $\mathcal{B}$ are not reached by all possible products in (2.1), and similarly the case that some elements in $\mathcal{A}$ are not reached by all possible products in (2.4). However, we will not discuss these aspects in this section, and they will not influence the general discussion below.

Now we can introduce the basis of the elements in $\mathcal{A}$, $\mathcal{B}$ for the extension of the 3-algebra into the one with integer mode numbers:

$$\mathcal{A}:\quad \{T^a_m\} \tag{2.5}$$

$$\mathcal{B}:\quad \{V^{ab}_m,\ C_{m,n}\} \tag{2.6}$$

where a, b are gauge indices, m, n are integers, which are mode numbers. Fixing all the modes to 0, we get back the ordinary 3-algebra with zero-mode generators.

We postulate a realization of the algebraic relations (2.1), (2.2), (2.4) as:

$$\left\langle T^a_m, T^b_n\right\rangle = V^{ab}_{m+n} + h^{ab}C_{m,n} \tag{2.7}$$

$$(V^{ab}_m, T^c_n) = f^{abc}_d T^d_{m+n} \tag{2.8}$$

$$(C_{m,n}, T^c_l) = g_{mn,lp}T^c_p \tag{2.9}$$

$$[T^a_m, T^b_n, T^c_l] = f^{abc}_d T^d_{m+n+l} + h^{ab}g_{mn,lp}T^c_p + h^{bc}g_{nl,mp}T^a_p + h^{ca}g_{lm,np}T^b_p \tag{2.10}$$

h^{ab} and f^{abc}_d are the metric and structure constants in the ordinary 3-algebra. The invariance of them demands respectively that $f^{abcd} = f^{abc}_e h^{ed}$ is totally antisymmetric and f^{abc}_d satisfies the fundamental identity. V^{ab}_m is antisymmetric in a, b, and $C_{m,n}$ is antisymmetric in m, n, while h^{ab} is symmetric in a, b. From (2.9), we see that $g_{mn,lp}$ is antisymmetric in m, n.

We need to check the Jacobi identities for the above assumed algebra. We first check the identity:

$$\left\langle (V^{ab}_m, T^c_n), T^d_l\right\rangle - \left\langle (V^{ab}_m, T^d_l), T^c_n\right\rangle = [V^{ab}_m, \left\langle T^c_n, T^d_l\right\rangle] \tag{2.11}$$

If we use the relations (2.7), (2.8), (2.9), we find that the above is equivalent to

$$[V^{ab}_m, C_{n,l}] = 0 \tag{2.12}$$

$$[V^{ab}_m, V^{cd}_{n+l}] = f^{abc}_e V^{ed}_{m+n+l} - f^{abd}_e V^{ec}_{m+n+l} + f^{abcd}(C_{m+n,l} + C_{m+l,n}) \tag{2.13}$$

Further if we set $l = 0$, due to antisymmetry property in m, n, we see that

$$C_{p,0} = 0 \tag{2.14}$$

for any integer p. So (2.12), (2.13) become simplified to

$$[V^{ab}_m, C_{n,l}] = 0 \tag{2.15}$$

$$[V^{ab}_m, V^{cd}_n] = f^{abc}_e V^{ed}_{m+n} - f^{abd}_e V^{ec}_{m+n} + f^{abcd}C_{m,n} \tag{2.16}$$

In appendix A, we have shown that the fV terms on the right hand side of (2.16) are in fact antisymmetric under the exchange of abm, cdn pairs, by virtue of the fundamental identity, and by using that f^{abc}_d furnishes a faithful and matrix representation of $(V_0^{ab})^c_d$.

We then check the Jacobi identity:

$$[V_l^{fg},[V_m^{ab},V_n^{cd}]] = [[V_l^{fg},V_m^{ab}],V_n^{cd}] - [[V_l^{fg},V_n^{cd}],V_m^{ab}] \tag{2.17}$$

We use the relations in (2.16) to simplify the above. The above is equivalent to two equations. One equation with the V terms are satisfied due to the fundamental identity for the structure constant f^{abc}_d.

The other equation imposes restrictions on $C_{m,n}$.

$$\begin{aligned}(f^{abc}_e f^{fged} - f^{abd}_e f^{fgec})C_{l,m+n}\\ = (f^{fgb}_e f^{cdea} - f^{fga}_e f^{cdeb})C_{n,l+m} + (f^{fgc}_e f^{abed} - f^{fgd}_e f^{abec})C_{m,l+n}\end{aligned} \tag{2.18}$$

The coefficients for $C_{l,m+n}$ and $C_{m,l+n}$ are negative with respect to each other. This is because of the identities

$$f^{abd}_e f^{fgec} = -f^{abd}_e f^{fgc}_{e'} h^{ee'} \tag{2.19}$$

$$f^{fgc}_e f^{abed} = -f^{abd}_{e'} f^{fgc}_e h^{ee'} \tag{2.20}$$

Thereby

$$f^{abd}_e f^{fgec} = f^{fgc}_e f^{abed} \tag{2.21}$$

since the metric $h^{ee'}$ is symmetric, and similarly for another term. The derivation in (2.19), (2.20), (2.21) only assumes that the metric h^{ab} is symmetric and is independent of the signature of the metric. So this holds for any signature, including Euclidean, Minkowski signatures and the case when there are zero eigenvalues in the metric.

The coefficients for $C_{l,m+n}$ and $C_{n,l+m}$ are also negative with respect to each other, and this is because

$$\begin{aligned}f^{abc}_e f^{fged} - f^{fgec} f^{abd}_e - f^{fga}_e f^{cdeb} + f^{fgb}_e f^{cdea}\\ = (f^{abc}_e f^{fge}_{d'} - f^{fgc}_e f^{abe}_{d'} - f^{fga}_e f^{bce}_{d'} - f^{fgb}_e f^{cae}_{d'})h^{dd'} = 0\end{aligned} \tag{2.22}$$

This is zero since it is the fundamental identity in the bracket contracted with the metric. In the above derivation we also only used that the metric is symmetric and the derivation is independent of the signature of the metric.

Thereby we have

$$C_{l,m+n} + C_{n,l+m} + C_{m,n+l} = 0 \tag{2.23}$$

This relation implies the recursion relation

$$C_{m,k-m} = C_{m-1,k-m+1} + C_{1,k-1} \tag{2.24}$$

Using this we get $C_{2,k-2} = 2C_{1,k-1}$, and using this recursion relation (2.24) $m-1$ times we get

$$C_{m,k-m} = mC_{1,k-1} \tag{2.25}$$

Thereby

$$kC_{1,k-1} = 0 \tag{2.26}$$

So we have $C_{m,k-m} = m\delta_{k,0}C_{1,k-1}$, or equivalently

$$C_{m,n} = m\delta_{m,-n}C_{1,-1} \tag{2.27}$$

We see that $C_{1,-1}$ is the only independent non-zero central charge, and we may define

$$C \equiv C_{1,-1} \tag{2.28}$$

Then (2.16) is simplified to (see also [7] without the C term)

$$[V_m^{ab}, V_n^{cd}] = f_e^{abc}V_{m+n}^{ed} - f_e^{abd}V_{m+n}^{ec} + f^{abcd}m\delta_{m,-n}C \tag{2.29}$$

By using the analysis in appendix A, we can rewrite the above in a way that is manifestly antisymmetric under the exchange of abm, cdn pairs:

$$[V_m^{ab}, V_n^{cd}] = \frac{1}{2}(f_e^{abc}V_{m+n}^{ed} - f_e^{abd}V_{m+n}^{ec} + f_e^{cdb}V_{m+n}^{ea} - f_e^{cda}V_{m+n}^{eb}) + f^{abcd}m\delta_{m,-n}C \tag{2.30}$$

(2.29) and (2.30) are equivalent modulo the fundamental identity, see appendix A. We then need to check (2.17) for the new expression (2.30), and we find that the equation with the C terms yields the same equation, and the equation with the V terms again satisfies, by using the fundamental identity multiple times.

Now we see that $g_{mn,lp}$ and (2.9) are simplified to

$$g_{mn,lp} = m\delta_{m,-n}\ g_{l,p} \tag{2.31}$$
$$(C, T_l^c) = g_{l,p}T_p^c \tag{2.32}$$

where $g_{l,p}$ is a function of l and p. We have not assumed any symmetry property for $g_{l,p}$.

We next look at the Jacobi identity:

$$\left\langle (C, T_n^c), T_l^d \right\rangle - \left\langle (C, T_l^d), T_n^c \right\rangle = [C, \left\langle T_n^c, T_l^d \right\rangle] \tag{2.33}$$

This identity is equivalent to two equations, one for the V terms, and another for the C terms:

$$g_{n,p}V_{p+l}^{cd} + g_{l,q}V_{q+n}^{cd} = 0 \tag{2.34}$$
$$g_{n,p}\delta_{p,-l}p - g_{l,q}\delta_{q,-n}q = 0 \tag{2.35}$$

where p or q is summed over. By just looking at the case $l = n$ for the first equation, we infer

$$g_{l,p} = 0 \tag{2.36}$$

So far, the rest of the Jacobi identities involve two elements in $\mathcal{B}$, and one elements in $\mathcal{A}$, and is equivalent to an identity of five elements in $\mathcal{A}$. This equation, in the present case, is the fundamental identity for the 3-bracket algebra (2.10), and since $g_{l,p} = 0$, or $g_{mn,lp} = 0$, this is the same as the fundamental identity for the structure constant f_d^{abc}.

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To summarize, the extension with mode numbers, under various consistency conditions[1] and assuming the ansatz (2.7)–(2.10), is

$$\left\langle T^a_m, T^b_n \right\rangle = V^{ab}_{m+n} + h^{ab} m \delta_{m,-n} C \tag{2.37}$$
$$(V^{ab}_m, T^c_n) = f^{abc}_d T^d_{m+n} \tag{2.38}$$
$$(C, T^c_l) = 0 \tag{2.39}$$
$$[V^{ab}_m, V^{cd}_n] = \frac{1}{2}(f^{abc}_e V^{ed}_{m+n} - f^{abd}_e V^{ec}_{m+n} + f^{cdb}_e V^{ea}_{m+n} - f^{cda}_e V^{eb}_{m+n}) + f^{abcd} m \delta_{m,-n} C \tag{2.40}$$
$$[V^{ab}_m, C] = 0 \tag{2.41}$$
$$[T^a_m, T^b_n, T^c_l] = f^{abc}_d T^d_{m+n+l} \tag{2.42}$$

This algebra has various subalgebras. If we look at the generators with zero modes, i.e. if we truncate the algebra keeping only the modes $m, n, l = 0$, we get the ordinary 3-algebra. This extended algebra of course includes the infinite dimensional extension of the $so(4)$ 3-algebra and the direct sum of the $so(4)$ 3-algebras, by adding mode numbers to each generators. The central charge C appears on the right hand sides of (2.40) and (2.37), and may introduce normal ordering issues in the products of operators.

If we start from (2.40), we can look at the subalgebra by fixing $a = c = *$, where $*$ is a specified gauge index, we get (see also [7])

$$[V^{*b}_m, V^{*d}_n] = f^{*bd}_e V^{*e}_{m+n} \tag{2.43}$$

which is a Lie algebra, and the Jacobi identity for the 3-index structure constant f^{*bd}_e, that is $f^{*[bd}_e f^{g]e*}_h = 0$, is a component equation of the fundamental identity for the 4-index structure constant f^{abd}_e, and is satisfied as long as the fundamental identity is satisfied.

Under the truncation (2.43), the central charge C disappears on the right hand side, due to the total antisymmetry of f^{abcd} in the last term of (2.40), thereby this extension is not equivalent to the usual infinite dimensional extension of Lie algebras with central charges, and is intrinsically 3-algebraic. This also means that the effects of C may not be seen after taking the limit to a D2-brane gauge theory. We also mention if we hypothetically had a term

$$g^{abcd} m \delta_{m,-n} C \tag{2.44}$$

where

$$g^{abcd} = h^{bc} h^{ad} - h^{ac} h^{bd} \tag{2.45}$$

on the right hand side of (2.40), we could have kept the C charge on the right hand side of (2.43), but this term (2.44) will not satisfy the Jacobi identities, primarily due to that g^{abcd} is not totally antisymmetric, in contract with f^{abcd}.

[1]More analysis on the manifest antisymmetry under the exchange of abm, cdn pairs in (2.40) is in appendix A. We used expression (2.30) instead of (2.29) in (2.40).

2.2 Infinite dimensional extensions with a Lorentzian metric

In this subsection, we discuss a different infinite dimensional extension of the 3-algebra, that is different from the ansatz (2.7)–(2.10) used in subsection 2.1, and we focus on the algebra with a metric of Minkowski or Lorentzian signature.

If we consider the 3-bracket algebra with a Lorentzian metric [23 – 25], we may start from the ansatz in [27],

$$[T^a, T^b, T^c] = tr(T^a)[T^b, T^c] + tr(T^b)[T^c, T^a] + tr(T^c)[T^a, T^b] + T^- tr(T^a, [T^b, T^c]) \quad (2.46)$$

where T^- is a central element in the 3-bracket algebra. This ansatz will be equivalent to [23 – 25] if we single out an identity matrix $\frac{1}{N}\mathbf{1}$ and make other T^as traceless.

We may directly start from a standard KM algebra for a Lie algebra,

$$[T^a_m, T^b_n] = \lambda^{ab}_c T^c_{m+n} + h^{ab} m \delta_{m,-n} T^- \quad (2.47)$$

$$[T^a_m, T^-] = 0 \quad (2.48)$$

We can plug these 2-brackets into[2] the defining equation for the 3-brackets in (2.46), and then we have

$$\begin{aligned}
[T^a_m, T^b_n, T^c_l] &= \lambda^{bc}_d tr(T^a_m) T^d_{n+l} + \lambda^{ca}_d tr(T^b_n) T^d_{l+m} + \lambda^{ab}_d tr(T^c_l) T^d_{m+n} + \lambda^{abc}\delta_{m+n+l,0} T^- \\
&\quad + \{h^{bc} tr(T^a_m) n\delta_{n,-l} + h^{ca} tr(T^b_n) l \delta_{l,-m} + h^{ab} tr(T^c_l) m \delta_{m,-n}\} T^- \quad (2.49)\\
[T^+, T^a_m, T^b_n] &= \lambda^{ab}_c tr(T^+) T^c_{m+n} \quad (2.50)\\
[T^-, T^a_m, T^b_n] &= 0 \quad (2.51)
\end{aligned}$$

in which we used the metric of the 3-algebra, and T^+ is another null generator in the Lorentzian 3-algebra. Two $h^{ab} tr(T^+) m \delta_{m,-n} T^-$ terms with opposite signs in (2.50) are cancelled. Other brackets are zero.

In this case, the fundamental identity for the 3-brackets (2.46) will be satisfied, if the Jacobi identities for the 2-brackets are satisfied [7, 15, 28]. This is indeed the case since the Jacobi identities for (2.47)–(2.48) are satisfied. If we set $m = n = 0$, the equation (2.50) defines the Lie algebra where $[T^+, \cdot\,, \,\cdot]$ defines the Lie algebra commutator, and if we keep the general m, n, (2.50) also defines an ordinary KM algebra but with the central term disappeared, similar to the discussion in subsection 2.1.

If we make the T^a_ms traceless, and make a redefinition $T^+ \to kT^+$, where $tr(T^+) = k \neq 0$, then the algebra becomes simplified:

$$[T^a_m, T^b_n, T^c_l] = \lambda^{ab}_c \delta_{m+n+l,0} T^- \quad (2.52)$$

$$[T^+, T^a_m, T^b_n] = \lambda^{ab}_c T^c_{m+n} \quad (2.53)$$

$$[T^-, T^a_m, T^b_n] = 0 \quad (2.54)$$

All 6 types of fundamental identities are satisfied. This can be viewed as an infinite dimensional extension of the Lorentzian algebra [23 – 25], ([26, 27, 7]), and reduces to the

[2]We thank Andreas Gustavsson for making a suggestion of this different type of extension.

latter when keeping $m = n = l = 0$. The T^- generator in the Lorentzian algebra can thereby have an interpretation of a central term in a underlying KM algebra (2.47).

We may look for a generating function for the generators with different modes, for example, if we look at the T^a_m generators, we may derive them from the expansion of

$$T^a(\sigma) = \frac{1}{2\pi}\sum_m T^a_m e^{im\sigma} \tag{2.55}$$

where σ is periodic with periodicity 2π, and $m \in \mathbf{Z}$. The generating functions $T^a(\sigma)$ may be viewed as valued in an internal direction σ. This may be relevant if the world volume fields on the multiple M2-branes carry not only gauge indices and Lorentz indices, but also internal indices corresponding to boundary lines of open membranes stretching between M2-branes. This is also relevant for the explanation of the M2 to D2 reduction. In the above assumption, this limit may involve integrating the σ circle, when one of the transverse scalars has an abelian component which under a gauge choice is identified with the σ circle, and receives a periodicity and is then integrated out. The above assumption seems to be rather natural in explaining the appearance of the periodicity.

It would be interesting to understand the relevance and the problem of the classification of the physical unitary representations of such algebras, especially the one for the $so(4)$ 3-algebra and the direct sum of the $so(4)$ 3-algebras, as well as the Lorentzian 3-algebra. In subsection 3.2, we also emphasize that a Lagrangian 2-tensor naturally lives in the algebra of $\mathcal{B}$.

3. Extensions with different signatures of the metric

3.1 Algebras with different signatures

In this section, we first study the algebra with different signatures of the metric, with the motivation of embedding a general Lie algebra, including the case of semisimple Lie algebras and the case of their direct sum with abelian ones. We consider both the cases when the metric has a negative eigenvalue and when it has a zero eigenvalue.

If we want to form a Lie subalgebra, we may pick a index $+$, similar to the relation in (2.43) when we pick a index $*$, so that

$$f^{+ab}_c = \lambda^{ab}_c \tag{3.1}$$

where λ^{ab}_c is a Lie algebra structure constant.

In this case, the covariant derivative

$$D_\mu X^I_a = \partial_\mu X^I_a - f^{dbc}{}_a A_{\mu cd} X^I_b \tag{3.2}$$

contains a piece

$$\partial_\mu X^I_a - \lambda^{bc}_a A'_{\mu c} X^I_b \tag{3.3}$$

where $A'_{\mu c} = 2A_{\mu c+}$, which looks the same as in a conventional gauge theory.

However we do not want T^+ to appear also on the right hand side of the 3-brackets, since if that is the case we will have a very strong Plucker type relation

$$\lambda^{cde}\lambda^{ab}_{g} = \lambda^{ab[c}\lambda^{de]}_{g} \tag{3.4}$$

from the fundamental identity, when checking $[T^a, T^b, [T^c, T^d, T^e]]$. This identity will only allow $so(3)$, direct sum of $so(3)$s, and the direct sum of them with $u(1)$s, as solutions. We want to avoid this identity so we let

$$f^{abc}_{+} = 0 \tag{3.5}$$

Then we need to check the total antisymmetry of f^{+abc} :

$$\begin{aligned} f^{+abc} &= f^{+ab}_{c} \\ &= \lambda^{ab}_{c} \end{aligned} \tag{3.6}$$

$$\begin{aligned} f^{abc+} &= f^{abc}_{+}h^{++} + f^{abc}_{-}h^{-+} \\ &= f^{abc}_{-}h^{-+} \\ &= -\lambda^{ab}_{c} \end{aligned} \tag{3.7}$$

where we used that the metric in the Lie algebra subspace is Euclidean.

Since $f^{abc}_{-}h^{-+}$ is non-zero, we infer that we must pick another generator T^- which has mixing with T^+ in the metric. We want to check the total antisymmetry of f^{-abc} :

$$f^{-abc} = f^{-ab}_{c} \tag{3.8}$$

$$\begin{aligned} f^{abc-} &= f^{abc}_{-}h^{--} + f^{abc}_{+}h^{-+} \\ &= f^{abc}_{-}h^{--} \\ &= -f^{-ab}_{c} \end{aligned} \tag{3.9}$$

We can rotate the subspace of T^- and T^+, so there is no need to put f^{-ab}_{c} as another copy of the Lie algebra structure constant, since we can redefine T^- and T^+ by $T^- - T^+$ and $T^- + T^+$. Because of this symmetry, we can choose that f^{+ab}_{c} gives the Lie algebra structure constant, while making

$$f^{-ab}_{c} = 0 \tag{3.10}$$

From the first derivation in (3.7) we know $f^{abc}_{-} \neq 0$, thereby from (3.9) we see

$$h^{--} = 0 \tag{3.11}$$

Without loss of generality we can choose

$$f^{abc}_{-} = \lambda^{ab}_{c} \tag{3.12}$$

$$h^{-+} = -1 \tag{3.13}$$

from (3.7). If we choose opposite signs for f^{abc}_{-} and h^{-+}, this would be equivalent to redefining T^- as $-T^-$, so this sign option is not necessary.

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The total antisymmetry of the $f^{ab+-} = 0$ is trivially satisfied in this algebra, and we have assumed that there is no mixing of metric between the $+,-$ subspace and the a, b subspace.

We look at the determinant of the metric

$$\det h = h^{--}h^{++} - (h^{-+})^2 = -1 \tag{3.14}$$

Now we look at the value of h^{++}. The value of it will not change the $\det h = -1$. Thereby there is still a symmetry. This value can be shifted away by redefining T^+ as

$$T^+ + \frac{1}{2}h^{++}T^- \tag{3.15}$$

which completely fixed that symmetry. Now the new T^+ has metric

$$h^{++} = 0 \tag{3.16}$$

which is a simplified choice.

Thereby for this algebra, the bracket $[T^+,\ \cdot,\ \cdot]$ defines the Lie algebra commutator. The fundamental identity is satisfied due to the Jacobi identity of the Lie algebra structure constant, which is the only non-trivial identity for this case. This algebra has been obtained independently by [23 − 25] and independently by the author [26] before the appearance of [23 − 25]. In the above, we present a modest derivation, with the new emphasis that this embedding is a very rare solution to the fundamental identity and does not admit obvious alternatives. The above eq. (3.4) would also imply that we can add at most products of $so(3)$s or abelian ones to the Lorentzian 3-algebra. The above derivation also makes a preparation for the discussion below in the case of a zero eigenvalue in the metric.

Now we discuss the situation when there is a zero eigenvalue in the metric, for example if the metric has the signature $(0, +, +, \ldots, +)$. We denote the null generator as T^0. So we have

$$h^{00} = 0, \qquad h^{ab} = \delta^{ab} \tag{3.17}$$

We want to consider the value of f_c^{0ab}. If we make this as a structure constant of a Lie algebra, like (3.1), then in order to avoid the strong relation in (3.4), we need another null generator, which has mixing with T^0 in the metric, see e.g. (3.7). This goes back to the $\det h = -1$ case in the previous discussion. So we would try to make simply

$$f_c^{0ab} = 0 \tag{3.18}$$

However, we can still make f_0^{abc} as a structure constant λ_c^{ab} of a Lie algebra, without violating any constraints. Thereby we have the simple algebra

$$[T^a, T^b,\ T^c] = \lambda_c^{ab}T^0 \tag{3.19}$$

$$[T^0, T^a,\ T^b] = 0 \tag{3.20}$$

The metric invariance is satisfied since

$$f_0^{abc} = \lambda_c^{ab}, \qquad f_c^{ab0} = 0, \qquad f_d^{abc} = 0 \tag{3.21}$$

$$f^{abc0} = 0, \qquad f^{abcd} = 0 \tag{3.22}$$

The fundamental identity is also satisfied.

The theory corresponding to this algebra (3.19), (3.20), (3.17) will be studied in the second part of the subsection 3.2. It is much less appealing than the det $h = -1$ case, however it has an advantage that there is no any negative components in the metric, and the resulting theory is manifestly unitary.

3.2 Lagrangians with different signatures

The Lagrangian of the corresponding theory was derived by first obtaining the on-shell equations of motion, after examining the closure of the supersymmetry algebra in [1–3], and later contracted with a metric. We may write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}_{ab} h^{ab} \tag{3.23}$$

$\mathcal{L}$ must be invariant under gauge transformations. In the component form, $\mathcal{L}_{ab}$ is

$$\begin{aligned}\mathcal{L}_{ij} = & -\frac{1}{2}(\partial_\mu X_i^I - \tilde{A}_\mu{}^b{}_i X_b^I)(\partial^\mu X_j^I - \tilde{A}^{\mu b}{}_j X_b^I) + \frac{i}{2}\bar{\Psi}_i \Gamma^\mu (\partial_\mu \Psi_j - \tilde{A}_\mu{}^b{}_j \Psi_b) \\ & -\frac{1}{12} f_i^{abc} f_j^{efg} X_a^I X_b^J X_c^K X_e^I X_f^J X_g^K + \frac{i}{4} f_i^{abc} \bar{\Psi}_b \Gamma_{IJ} X_c^I X_j^J \Psi_a \\ & +\frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f_i^{abc} A_{\mu ab} \partial_\nu A_{\lambda cj} + \frac{2}{3} f_i^{cdb} f_j^{efa} A_{\mu ab} A_{\nu cd} A_{\lambda ef}\right)\end{aligned} \tag{3.24}$$

where we used i, j indices in place of a, b for clarity purpose, and the gauge connection is $(\tilde{A}_\mu)_i^a = (A_\mu)_{cd} f_i^{cda}$. In this component form, the structure constants only appear with 3 upper indices and 1 lower indices, and the gauge indices in the fields $X_a^I, \Psi_a, A_{\mu ab}$ only appear as lower gauge indices, so we have not used the metric yet and this expression is independent of the metric choice h^{ij}. So far the only assumption on the metric is that it is symmetric and gauge invariant.

The first term in the third line of (3.24) may be replaced by the term

$$+\frac{1}{2}\varepsilon^{\mu\nu\lambda} f_i^{abc} A_{\mu cj} \partial_\nu A_{\lambda ab} \tag{3.25}$$

since they differ by a total derivative term which may not be important for the theory defined on $R^{2,1}$ with no boundaries.

We may look at a gauge invariant 2-tensor

$$\mathcal{L}^{ab} = h^{ab} \mathcal{L} \tag{3.26}$$

This is gauge invariant since both h^{ab} and $\mathcal{L}$ are gauge invariant, and $\mathcal{L}^{ab}$ is an element in the algebra $\mathcal{B}$, as discussed in section 2. In other words,

$$[V_0^{cd},\ \mathcal{L}^{ab}] = 0, \qquad [C,\ \mathcal{L}^{ab}] = 0 \tag{3.27}$$

where V_0^{cd} is $V_{m=0}^{cd}$, the zero-mode generators discussed in section 2, and is an arbitrary gauge transformation.

It is interesting to note that $(\tilde{F}_{\mu\nu})_a^b$ is in the algebra $\mathcal{B}$, and the on-shell equation of motion [1]–[3] relates it to

$$(\tilde{F}_{\mu\nu})_a^b = -\epsilon_{\mu\nu\lambda} f_a^{cdb}(X_c^J D^\lambda X_d^J + \frac{i}{2}\bar{\Psi}_c \Gamma^\lambda \Psi_d) \tag{3.28}$$

which means that $(\tilde{F}_{\mu\nu})_a^b$ contains no new independent degrees of freedom, and is, on-shell, the Hodge dual of the bilinear current of the scalars and spinors. This equation is intimately related to that the self-coupling of the gauge fields is of the Chern-Simons type [4] or the like. Both sides of (3.28) are the sources coupled to the gauge fields. Moreover, since the X_d^J and Ψ_d fields live in the algebra $\mathcal{A}$, this equation is also very supportive of the view that elements in $\mathcal{B}$ are formed by anti-symmetric bilinear products of the elements in $\mathcal{A}$.

In the rest of this subsection, we discuss the theory for the case when there is a zero eigenvalue in the metric, as in the algebra (3.19), (3.20), (3.17) in subsection 3.1. In this case $h^{00}\mathcal{L}_{00} = 0$, so $\mathcal{L}_{00}$ does not contribute to the Lagrangian $\mathcal{L}$. However, $\mathcal{L}_{00}$ has its own equations of motion. Let's discuss the equations of motion corresponding to $\mathcal{L}_{00} = \mathcal{L}'$.

$$\begin{aligned}\mathcal{L}' = & -\frac{1}{2}(\partial_\mu X_0^I - \tilde{A}_\mu{}^b{}_0 X_b^I)(\partial^\mu X_0^I - \tilde{A}^{\mu b}{}_0 X_b^I) + \frac{i}{2}\bar{\Psi}_0 \Gamma^\mu(\partial_\mu \Psi_0 - \tilde{A}_\mu{}^b{}_0 \Psi_b) \\ & + \frac{i}{4} f_0^{abc} \bar{\Psi}_b \Gamma_{IJ} X_c^I \Psi_a X_0^J - \frac{1}{12} f_0^{abc} f_0^{efg} X_a^I X_b^J X_c^K X_e^I X_f^J X_g^K \\ & + \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(\tilde{A}_\mu{}^a{}_0 \partial_\nu A_{\lambda a0} + \frac{2}{3} A_{\mu ab} \tilde{A}_\nu{}^b{}_0 \tilde{A}_\lambda{}^a{}_0\right)\end{aligned} \tag{3.29}$$

where $(\tilde{A}_\mu)_0^a = (A_\mu)_{cd} f_0^{cda}$. Again, the first term in the last line of (3.29) can be replaced by

$$+\frac{1}{2}\varepsilon^{\mu\nu\lambda} A_{\mu a0} \partial_\nu \tilde{A}_\lambda{}^a{}_0 \tag{3.30}$$

up to a total derivative term.

The equations of motion are

$$D^2 X_0^I - \frac{i}{2}\bar{\Psi}_c \Gamma^I_J X_d^J \Psi_b f_0^{cdb} = 0 \tag{3.31}$$

$$\Gamma^\mu D_\mu \Psi_0 + \frac{1}{2}\Gamma_{IJ} X_c^I X_d^J \Psi_b f_0^{cdb} = 0 \tag{3.32}$$

$$(\tilde{F}_{\mu\nu})_0^b = -\epsilon_{\mu\nu\lambda}\left(X_c^J D^\lambda X_d^J + \frac{i}{2}\bar{\Psi}_c \Gamma^\lambda \Psi_d\right) f_0^{cdb} \tag{3.33}$$

The susy transformations and gauge transformations are

$$\delta X_0^I = i\bar{\epsilon}\Gamma^I \Psi_0 \tag{3.34}$$

$$\delta \Psi_0 = (\partial_\mu X_0^I - \tilde{A}_\mu{}^b{}_0 X_b^I)\Gamma^\mu \Gamma_I \epsilon - \frac{1}{6} X_b^I X_c^J X_d^K f_0^{bcd} \Gamma_{IJK}\epsilon \tag{3.35}$$

$$\delta(\tilde{A}_\mu)_0^a = i\bar{\epsilon}\Gamma_\mu \Gamma_I X_c^I \Psi_d f_0^{cda} \tag{3.36}$$

and

$$\delta X_0^I = \tilde{\Lambda}_0^b X_b^I \tag{3.37}$$

$$\delta \Psi_0 = \tilde{\Lambda}_0^b \Psi_b \tag{3.38}$$

$$\delta(\tilde{A}_\mu)_0^b = D_\mu \tilde{\Lambda}_0^b \tag{3.39}$$

where $\tilde{\Lambda}^a_0 = \Lambda_{cd} f^{cda}_0$.

We may view that $\mathcal{L}'$ gives a certain theory by itself, which is a gauge theory with the gauge connection $(\tilde{A}_\mu)^a_0$. $\mathcal{L}'$ is gauge invariant under the gauge transformation corresponding to this connection. This theory is decoupled with the theory given by $\mathcal{L}$, which is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu X^{aI}\partial^\mu X^I_a + \frac{i}{2}\bar{\Psi}^a\Gamma^\mu\partial_\mu\Psi_a \tag{3.40}$$

These are free theories, with global symmetry given by the Lie algebra associated with f^{abc}_0. The susy and local gauge transformations are respectively

$$\delta X^I_a = i\bar{\epsilon}\Gamma^I\Psi_a\,, \qquad \delta\Psi_a = \partial_\mu X^I_a\Gamma^\mu\Gamma_I\epsilon \tag{3.41}$$

$$\delta X^I_a = 0, \qquad \delta\Psi_a = 0 \tag{3.42}$$

since $\tilde{\Lambda}^a_b = 0$.

It has global symmetry transformations associated with the Lie algebra,

$$\delta X^I_a = \overline{\Lambda}^b_a X^I \tag{3.43}$$

$$\delta\Psi_a = \overline{\Lambda}^b_a \Psi_b \tag{3.44}$$

where $\overline{\Lambda}^b_a$ is a global gauge transformation parameter.

This theory is not very appealing since it is a free theory with a Lagrangian $\mathcal{L}$ and a global symmetry, decoupled from another theory with a Lagrangian $\mathcal{L}'$ and a local gauge symmetry, albeit an abelian one. However the advantage is that there is no any negative metric component in the algebra and the theory is straightforwardly unitary.

We also remark that if X^I_a receives a vev, then ${\tilde{A}_\mu}{}^a{}_0$ gets a mass, and after integrating out this massive gauge field (similar to [9] or [12, 13, 10]), one obtains a dynamical Yang-Mills type term of the form, from (3.29)

$$-\frac{1}{4}F^a_{\nu\lambda 0}F^{\nu\lambda}_{a0} \tag{3.45}$$

which is however abelian.

4. Conclusions and discussion

We constructed infinite dimensional 3-algebras (2.37)–(2.42) corresponding to extending ordinary 3-algebras by adding mode numbers. The consistency conditions and Jacobi identities single out a unique central charge (2.28) that appears on the right hand side of the algebraic relations (2.40), (2.37). This may introduce new normal ordering issues in operator products. This effect may not be seen after the limit when the theory goes to D2-brane gauge theory, since this centrally-extended algebra is intrinsically 3-algebraic. We also present a different infinite dimensional extension (2.52)–(2.54) for the Lorentzian 3-algebras, and interpret one of the null generators as a central term in a underlying KM algebra (2.47)-(2.48). These extended generators may be expanded by the generating functions like (2.55). It would be nice to understand the relevance and the relation of the

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extended algebras with M2-branes, especially the open membranes or wrapped membranes, and the problem of the classification of unitary representations of these algebras, especially the ones with $so(4)$ 3-algebra and their direct sums, as well as the Lorentzian 3-algebras.

We also explored ordinary 3-algebras with different signatures of the metric, that is consistent with metric invariance and the fundamental identity. We revisited the algebras with a negative eigenvalue in the metric, (3.1)–(3.16), which were obtained by the authors of [23 − 26]. To avoid the problem of negative kinetic terms, we explored the algebras with a zero eigenvalue in the metric, and present the simple algebra in (3.19), (3.20), (3.17). This theory is manifestly unitary, and is a local abelian gauge theory with Lagrangian $\mathcal{L}'$ (3.29) decoupled with another global gauge theory with Lagrangian $\mathcal{L}$ (3.40). We also emphasized that the Lagrangian 2-tensor $\mathcal{L}^{ab}$ (3.26) lives naturally in the algebra $\mathcal{B}$, and is gauge invariant.

A particular interesting theory is the mass deformed M2-brane theory preserving $so(4) \times so(4)$ R-symmetries, with degenerate vacua corresponding to representations of $so(4)$ and new BPS states due to non-central charges in the Poincare superalgebra [33]. The Jacobi identity of supercharges are non-trivial as emphasized in [33, 31, 32] (see also the wonderful discussions in [30]), and should be checked independently, even after obtaining the supercharge anticommutators. The smooth 11 dimensional gravity duals of these multiple vacua [35],[33] not only predicts that the vacua structure can be described by fermion bands on a cylinder, but also that there is a duality between m fivebranes wrapping one S^3, each constructed by n M2-branes, and n fivebranes wrapping another dual $\widetilde{S}^3$, each constructed by m M2-branes. These vacua could be viewed as fuzzy S^3 vacua, e.g. [43]–[47, 5]. There are domain walls connecting between different $so(4)$ representations, e.g. [31]. This is very similar to the instantons connecting between different $so(3)$ representations in the plane-wave matrix model, and in the gravity dual it was found [36] that when the $so(3)$ representations are very close to each other, the tunneling is mediated by Euclidean brane processes, and in the case when the $so(3)$ representations are not close to each other, it was proposed [36] to be described by a non-perturbative tunneling of a quantum mechanical eigenvalue system. There are also bounce solutions studied recently [37]. It would be nice to understand the tunneling between different $so(4)$ representations from a gauge theoretical point of view, especially in the illuminating framework of [1 − 3].

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A. Proof of an identity and antisymmetry

In section 2.1 we have checked all the Jacobi identities, the upper antisymmetry of the structure constant f_d^{abc}, and the symmetry of the metric h^{ab}, and arrived at the general expression in (2.37)–(2.42), which is consistent with the above mentioned three consistency conditions. One thing remains is that (2.29) is not manifestly antisymmetric under the exchange of abm, cdn pairs as in V_m^{ab}, V_n^{cd}, although this antisymmetry property is obviously true for the $so(4)$ 3-algebra. In this appendix, we prove that this is not a problem, due to the fundamental identity. Our analysis agrees with similar analysis and conclusion in [7]. We may rewrite (2.29) in two ways, which we will shown to be equivalent:

$$[V_m^{ab}, V_n^{cd}] = f_e^{abc} V_{m+n}^{ed} - f_e^{abd} V_{m+n}^{ec} + f^{abcd} m\delta_{m,-n} C \tag{A.1}$$

$$-[V_n^{cd}, V_m^{ab}] = -f_e^{cda} V_{m+n}^{eb} + f_e^{cdb} V_{m+n}^{ea} + f^{abcd} m\delta_{m,-n} C \tag{A.2}$$

We should understand V_m^{ab} as operators acting on the linear combination of the generators T_l^c via the definition $(V_m^{ab},\ T_l^c)$ as in (2.8). If (A.1), (A.2) are equivalent, we must have

$$f_e^{bcd} V_{m+n}^{ea} = f_e^{abc} V_{m+n}^{ed} + f_e^{acd} V_{m+n}^{eb} + f_e^{adb} V_{m+n}^{ec} \tag{A.3}$$

to be true when acting on an arbitrary linear combination of the generators, e.g. $\alpha_g T_l^g$. We then would demand

$$f_e^{bcd}(V_{m+n}^{ea},\ T_l^g) = f_e^{abc}(V_{m+n}^{ed},\ T_l^g) + f_e^{acd}(V_{m+n}^{eb},\ T_l^g) + f_e^{adb}(V_{m+n}^{ec},\ T_l^g) \tag{A.4}$$

By using (2.8), this is simplified to

$$f_e^{bcd} f_h^{eag} T_{m+n+l}^h = (f_e^{abc} f_h^{edg} + f_e^{acd} f_h^{ebg} + f_e^{adb} f_h^{ecg}) T_{m+n+l}^h \tag{A.5}$$

This is true since the coefficients in front of T_{m+n+l}^h form the fundamental identity, thus this proves the equivalence of (A.1), (A.2) and the antisymmetry under the exchange of abm, cdn pairs in V_m^{ab}, V_n^{cd}. Q.E.D.

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